BEHAVIOR OF A RIGIDLY PLASTIC CYLINDRICAL SHELL EXPOSED TO INTERNAL PRESSURE

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A law for conservation of energy is used in solving a system of equations describing the one-dimensional motion of an ideally plastic incompressible shell exposed to an expanding polytropic gas in equilibrium. Analytic expressions are obtained for determining the stress and velocity fields in the shell as a function of the displacement of the internal shell boundary.

The behavior of an ideally plastic incompressible shell exposed to pressure somewhat exceeding the yield stress of the material was considered in [1-8]. The motion problem for an ideally plastic shell exposed to expanding detonation products in equilibrium lacks an analytic solution in the form of finite dependences on coordinate and time. However, it can be solved if we take the magnitude of the external (b) of internal (a) radius of the shell as the independent variable.

Let us consider in the figure a plane deformation of a cylindrical shell exposed to detonation products obeying the expansion law

$$pV^k=\mathrm{const}, \tag{1}$$

where p and V are the pressure and specific volume, respectively, of the detonation products. The stresses σ_r, σ_θ, and σ_z are principal. The internal and external initial shell radii are denoted by a_0 and b_0, respectively, and the current radii, by a and b.

We write the law of conservation of energy of the system in the form

$$E+W+E_f=E_0$$

or per unit of length

$$\tilde{E}+\tilde{W}+\tilde{E}_f=1, \tag{2}$$

where E_0 and E are the initial and current internal energies of the detonation products, respectively, W is the shell kinetic energy, and E_f is the work of plastic deformation per unit of length. The kinetic energy of the detonation can be neglected. We will consider in detail each term in Eq. (2).

1. Equation (1) implies that

$$p=p_0(a_0/a)^{2k}, \tag{3}$$

where $p_0 = \rho_0 D^2/8$ is the instantaneous detonation pressure, ρ_0 is the density of the explosive, and D is the rate of detonation.

The internal energy for an ideal gas is given by

$$E=pV/(\varkappa-1),$$

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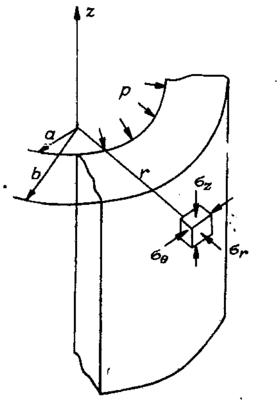

Fig. 1.

where V is the volume of the detonation products per unit of length, or using Eq. (3) and taking into account that

$$V=\pi a^2,$$

we obtain

$$E=\frac{\pi p_0 a_0^{2k}}{(k-1)\,a^{2(k-1)}}.$$

Since

$$E_0=\pi p_0 a_0^2/(k-1),$$

we have

$$\tilde{E}=E/E_0=(a_0/a)^{2(k-1)}. \tag{4}$$

2. The equation for the shell kinetic energy will be written in the form

$$W=\int_m (v^2/2)\,dm,$$

where $dm = 2\pi\gamma_0 r dr$; v is the radial velocity of shell particles, and γ_0 is the density of the shell material.

Assuming the shell material to be incompressible, we determine the integral of the continuity equation,

$$v=\dot{a}a/r, \tag{5}$$

where r is a Euler coordinate and $\dot{a} = da/dt$ is the speed of the interior surface of the shell.

Because of Eq. (5) we have

$$W=\pi\gamma_0 a^2\dot{a}^2\ln(b/a);$$
$$\tilde{W}=W/E_0=(k-1)\gamma_0\dot{a}^2(a/a_0)^2\ln(b/a)/p_0. \tag{6}$$

We denote by $\langle v\rangle$ the mean shell velocity, determined from the law of conservation of momentum,

$$\langle v\rangle M=\int_m v\,dm.$$

where M is shell mass per unit of length. The equation for relative kinetic energy takes the form

$$\tilde{W}=(k-1)\gamma_0[(b_0/a_0)^2-1]\langle v\rangle^2/2p_0. \tag{7}$$

3. We write the equation for the work of plastic deformation in the form

$$E_f=\int_U A_p dU;\quad A_p=\int_0^{\varepsilon_i}\sigma_i d\varepsilon_i,$$

where σ_i and ε_i are stress and deformation intensities, respectively, and U is shell volume per unit of length.

Taking the plasticity condition in the form

$$\sigma_i=\sqrt{3}\,\varkappa Y/2, \tag{8}$$

where Y is the dynamic yield stress ($\varkappa=2/\sqrt{3}$ for the Mises–Henke plasticity condition and $\varkappa = 1$ for the

Saint Venant–Tresk plasticity condition), and neglecting elastic deformations. We have

$$E_f = \sqrt{3}\,\varkappa Y \pi \int_a^b \varepsilon_i r\,dr. \tag{9}$$

We find the tangential logarithmic deformation

$$\varepsilon_\theta = -\ln(1-u/r) = -\ln\left[1-(a^2-a_0^2)/r^2\right]^{1/2},$$

where u is the radial displacement. When $\varepsilon_\theta = -\varepsilon_r$, it follows that

$$\varepsilon_i = -\frac{\sqrt{3}}{3}\ln\left(1-\frac{a^2-a_0^2}{r^2}\right).$$

Substituting ε_i in Eq. (9) and integrating, we obtain

$$E_f = \pi\varkappa Y\left[a^2\ln(b/a) + b_0^2\ln(b/b_0) - a_0^2\ln(b/a_0)\right];$$
$$\widetilde{E}_f = (k-1)\varkappa YA/p_0, \tag{10}$$

where $A = (a/a_0)^2\ln(b/a) + (b_0/a_0)^2\ln(b/b_0) - \ln(b/a_0)$. Substituting Eqs. (4), (6), and (10) in Eq. (2), we obtain

$$\dot{a} = \left[p_0\,\frac{1-(a_0/a)^{2(k-1)} - (k-1)\varkappa YA/p_0}{(k-1)\gamma_0 (a/a_0)^2\ln(b/a)}\right]^{1/2}. \tag{11}$$

The initial conditions are given by $a(0) = a_0$ and $\dot{a}(0) = 0$. Equation (11) together with Eq. (5) and the incompressibility condition $b^2 - a^2 = b_0^2 - a_0^2$ allows us to determine the mass velocity in any radial shell section as a function of the position of its internal boundary a.

To determine the stress state, we will use the Euler equation

$$\gamma_0\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) = \frac{\partial\sigma_r}{\partial r} - \frac{\sigma_\theta - \sigma_r}{r}. \tag{12}$$

Substituting Eq. (5) and the derivatives $\partial v/\partial t$ and $\partial v/\partial r$ in Eq. (12) and also using the plasticity condition of Eq. (8), we obtain

$$\frac{\partial\sigma_r}{\partial r} = \frac{\varkappa Y}{r} + \gamma_0\left(\frac{a\ddot{a} + \dot{a}^2}{r} - \frac{a^2\dot{a}^2}{r^3}\right), \tag{13}$$

where $\ddot{a} = d^2a/dt^2$ is the acceleration of the interior surface of the shell. Integrating Eq. (13) over r between a and the current value of r and taking into account the boundary condition on the internal surface, $\sigma_r = -p$ at $r = a$, we obtain the equation

$$\sigma_r = -p + \varkappa Y\ln\frac{r}{a} + \gamma_0\left(a\ddot{a} + \dot{a}^2\right)\ln\frac{r}{a} + \gamma_0\left(\frac{a^2\dot{a}^2}{2r^2} - \frac{\dot{a}^2}{2}\right), \tag{14}$$

which, when using the condition on the external boundary $\sigma_r = 0$ at $r = b$, takes the form

$$-p + \varkappa Y\ln\frac{b}{a} + \gamma_0\left(a\ddot{a} + \dot{a}^2\right)\ln\frac{b}{a} + \gamma_0\left(\frac{a^2\dot{a}^2}{2b^2} - \frac{\dot{a}^2}{2}\right) = 0. \tag{15}$$

Using Eqs. (11) and (15), we find in Eq. (4) the acceleration of the internal surface,

$$\ddot{a} = \frac{p_0(a_0/a)^{2k}}{a\gamma_0\ln(b/a)} - \frac{2\varkappa Y}{\sqrt{3}a\gamma_0} - p_0\frac{1-(a_0/a)^{2(k-1)} - (k-1)\varkappa YA/p_0}{(k-1)\gamma_0(a/a_0)^2\ln(b/a)}\left[\frac{1}{a} + \frac{a}{2b^2\ln(b/a)} - \frac{1}{2a\ln(b/a)}\right]. \tag{16}$$

Substituting Eqs. (11) and (16) in Eq. (14), we obtain

$$\sigma_r = p_0 \frac{(a_0/a)^{2k} \ln (r/b)}{\ln (b/a)} + p_0 \frac{1 - (a_0/a)^{2(k-1)} - (k-1) \varkappa Y A/p_0}{2(k-1)(a/a_0)^2 \ln (b/a)} \left[\left(\frac{a}{r}\right)^2 - 1 - \frac{\ln (r/a)}{\ln (b/a)} \left(\frac{a^2}{b^2} - 1\right) \right]. \quad (17)$$

Thus, if we know the distribution of the radial stress component across the shell, we may determine the two other components σ_θ and σ_z of the stress tensor, if we use the plasticity condition of Eq. (8) and the plane deformation condition $\varepsilon_z = 0$ (ε_z is the axial deformation component).

We will use Eqs. (2), (4), (7), and (10) to determine the mean shell velocity $\langle v \rangle$. Carrying out the transformations, we obtain

$$\langle v \rangle^2 = 2p_0 \frac{1 - (a_0/a)^{2(k-1)} - (k-1) \varkappa Y A/p_0}{(k-1) \gamma_0 [(b_0/a_0)^2 - 1]}. \quad (18)$$

Since $p_0 = \rho_0 D^2/8$ and setting $k = 3$, we have

$$\frac{\langle v \rangle^2}{D^2} = \frac{\beta}{8} \left[1 - \left(\frac{a_0}{a}\right)^4 \right] - 2\varkappa \frac{YA}{(b_0/a_0)^2 - 1},$$

where $\beta = \rho_0 a_0^2/[\gamma_0(b_0^2 - a_0^2)]$ is the ratio of explosive mass to shell mass.

The second term takes into account energy losses due to plastic deformation. We write this term in the form

$$v_p = \frac{2\varkappa Y \left[a^2 \ln (b/a) + b_0^2 \ln (b/b_0) - a_0^2 \ln (b/a_0) \right.}{\gamma_0 D^2 \left(b_0^2 - a_0^2\right)}. \quad (19)$$

Let us assume that the stress distribution is invariant throughout the shell thickness in the course of its expansion. This formally means that we must set $a = b$ and $a_0 = b_0$ in Eq. (19). Then Eq. (19) becomes an equality,

$$v_p^2 = \frac{2\varkappa Y}{\gamma_0 D^2} \ln \frac{a}{a_0}$$

and Eq. (18) takes the form

$$\frac{\langle v \rangle^2}{D^2} - \frac{\beta}{8} \left[1 - \left(\frac{a_0}{a}\right)^4 \right] - \frac{2\varkappa Y}{\gamma_0 D^2} \ln \frac{a}{a_0}. \quad (20)$$

If we take the Tresk condition as the plasticity condition, Eq. (20) becomes

$$\frac{\langle v \rangle^2}{D^2} = \frac{\beta}{8} \left[1 - \left(\frac{a_0}{a}\right)^4 \right] - \frac{2Y}{\gamma_0 D^2} \ln \frac{a}{a_0},$$

which was given in [6].

Calculations carried out using Eqs. (11), (16), and (17) and a numerical method [8] demonstrated that the results coincide to within a numerical approximation.

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